# Global Orbit Patterns for One Dimensional Dynamical Systems

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# Global Orbit Patterns for One Dimensional Dynamical Systems

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**Abstract**

In this article, we study the behaviour of discrete one-dimensional dynamical systems associated to functions on finite sets. We formalise the global orbit pattern formed by all the periodic orbits (gop) as the ordered set of periods when the initial value thumbs the finite set in the increasing order. We are able to predict, using closed formulas, the number of gop for the set $\mathcal{F}_N$ of all the functions on $X$. We also explore by computer experiments special subsets of $\mathcal{F}_N$ in which interesting patterns of gop are found.

## 1 Introduction

Dynamical systems $x_{n+1} = f_\lambda(x_n)$ associated to a function $f_\lambda : [a, b] \to [a, b]$ with $(a, b) \in \mathbb{R}^2$ on a real interval are very well-known. Their periodic orbits, if any, $x^p_{n+1} = f^p_\lambda(x_n)$ are an important feature. There is a growing interest in numerical analysis and industrial mathematics to study such systems, or systems in higher dimensions [5]. Very often, dynamical systems in several dimensions are obtained coupling one-dimensional ones and their properties are strongly linked.

Simple dynamical systems often involve periodic motion. Quasi-periodic or chaotic motion is frequently present in more complicated dynamical systems. The most famous theorem in this field of research is the Sharkovskii's theorem, which addresses the existence of periodic orbits of continuous maps of the real line into itself. This theorem was once proved toward the year 1962 and published only two years after, A.N. Sharkovskii [10].

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Mathematical results concerning periodic orbits are often obtained using sets of real numbers. However, most of the time, as the complex behaviour of chaotic dynamical systems is not explicitly tractable, mathematicians have recourse to computer simulations. The main question which arises then is: does these numerical computations are reliable?

As an example O.E. Lanford III [4] reports the results of some computer experiments on the orbit structure of the discrete maps on a finite set which arise when an expanding map of the circle is iterated "naively" on the computer.
Due to the discrete nature of floating points used by computers, there is a huge gap between these results and the theoretical results obtained when this map is considered on a real interval. (This gap can be narrowed in some sense (i.e. avoiding the collapse of periodic orbits) in higher dimensions when ultra weak coupling is used [6, 7].)

Nowadays the claim is to understand precisely which periodic orbit can be observed numerically in such systems. In a first attempt we study in this paper the orbits generated by the iterations of a one-dimensional system on a finite set $X_N$ with a cardinal $N$. The final goal of a good understanding of the actual behaviour of dynamical systems acting on floating numbers (i.e. the numbers used by computers) will be only reached when this first step will be achieved.

On finite set, only periodic orbits can exist. For a given function we can compute all the orbits, all together they form a global orbit pattern. We formalise such a gop as the ordered set of periods when the initial value thumbs the finite set in the increasing order. We are able to predict, using closed formulas, the number of gop for the set $\mathcal{F}_N$ of all the functions on $X$. We also explore by computer experiments special subsets of $\mathcal{F}_N$ in which interesting patterns of gop are found.

This article is organized as follows: in section 2 we display some examples already known of such computational discrepancies for the logistic and circle maps. In section 3 we introduce a new mathematical tool: the global orbit pattern, in order to describe more precisely the behaviour of dynamical systems on finite sets. In section 4 we give some closed formulas giving the cardinal of classes of gop of $\mathcal{F}_N$. In section 5 we study the case of sets of functions with local properties versus their gop, in order to show the significance of these new tools.

# 2 Computational discrepancies

## 2.1 Approximated logistic map

As an example of such collapsing effects, O.E. Lanford III, presents the results of a sampling study in double precision of a discretization of the logistic map

$$x_{n+1} = 1 - 2x_n^2 \tag{1}$$

which has excellent ergodic properties. The precise discretization studied is obtained by first exploiting evenness to fold the interval $[-1, 0]$ to $[0, 1]$, i.e replacing (1) by

$$x_{n+1} = |1 - 2x_n^2| \tag{2}$$

On $[0, 1]$ it is not difficult to see that the folded map has the same periods as the original one. The working interval is then shifted from $[0, 1]$ to $[1, 2]$ by translation, in order to avoid perturbation in numerical experiment, by the special value 0. Then the translated folded map is programmed in straightforward way.
Out of 1,000 randomly chosen initial points,

- 890, i.e., the overwhelming majority, converged to the fixed point corresponding to the fixed point-1 in the original representation (1),
- 108 converged to a cycle of period 3,490,273,
- the remaining 2 converged to a cycle of period 1,107,319.

Thus, in this case at least, the very long-term behaviour of numerical orbits is, for a substantial fraction of initial points, in flagrant disagreement with the true behaviour of typical orbits of the original smooth logistic map.

## 2.2 Circle approximated maps

In the same paper, O.E. Lanford III, studies very carefully the numerical approximation of the map

$$x_{n+1} = 2x_n + 0.5x_n(1 - x_n) \pmod 1 \quad 0 \le x \le 1 \tag{3}$$

It is perhaps better to think this map (see Figure 1) as acting on the unit interval with endpoints identified, i.e., on the circle. Note that $f'(x) \geq 1.5$ everywhere, so $f$ is strictly expanding in a particularly clean and simple sense. As a consequence of expansivity, this mapping has about the best imaginable ergodic properties :

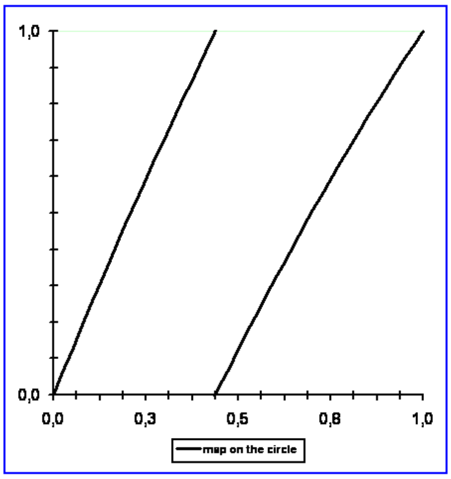


Figure 1: Graph of the map $f(x) = 2x + 0.5x(1 - x) \pmod 1$

- it admits a unique invariant measure $\mu$ equivalent to Lebesgue measure,
- the abstract dynamical system $(f, \mu)$ is ergodic and in fact isomorphic to a Bernoulli shift,
- a central limit theorem holds.

One consequence of ergodicity of $f$ relative to $\mu$ is that, for Lebesgue almost initial points $x$ in the unit interval, the corresponding orbit $f^{(n)}(x)$ is asymptotically distributed over the unit interval according to $\mu$.

In numerical experiments performed by the author, the computer working with fixed finite precision is able to represent finitely many points in the interval in question. It is probably good, for purposes of orientation, to think of the case where the representable points are uniformly spaced in the interval. The true smooth map is then *approximated* by a discretized map, sending the finite set of representable points in the interval to itself.

Describing the discretized mapping exactly is usually complicated, but it is *roughly* the mapping obtained by applying the exact smooth mapping to each of the discrete representable points and "rounding" the result to the nearest representable point.

O.E. Lanford III has done two kinds of experiments, in the first one he uses double precision floating points, in the second uniformly spaced points in the interval with several order of discretization (ranging from $2^{22}$ to $2^{25}$) are involved. In each experiment the questions addressed are:

| Double precision (sampling) 7 cycles found | | |
|---|---|---|
| period | basin size | relative size |
| 27,627,856 | 517 | 51.7 % |
| 88,201,822 | 320 | 32.0 % |
| 4,206,988 | 147 | 14.7 % |
| 4,837,566 | 17 | 1.7 % |
| 802,279 | 8 | 0.8 % |
| 6,945,337 | 6 | 0.6 % |
| 2,808;977 | 1 | 0.1 % |

Table 1: Coexisting periodic orbits of mapping (4) for double precision numbers.

- how many periodic cycles are there and what are their periods ?
- how large are their respective basins of attraction, i.e., for each periodic cycle, how many initial points give orbits with eventually land on the cycle in question?

On one hand, for the experiments using ordinary (IEEE-754) double precision - so that the working interval contains of the order of $10^{16}$ representable points - 1,000 initial points at random are used in order to sample the orbit structure, determining the cycles to which they converge.

The computations were accomplished by shifting the working interval from the initial interval [0, 1] to [1, 2] by translation, for the same previous reason as for the logistic map, i.e., the map actually iterated was

$$x_{n+1} = 2x_n + 0.5x_n(x_n - 1)(2 - x_n) \pmod 1 \quad 1 \leq x \leq 2 \tag{4}$$

Some results are displayed in Table 1.

On another hand, for relatively coarse discretizations the orbit structure is determined completely, i.e., all the periodic cycles and the exact sizes of their basins of attraction are found. Some representative results are given in Tables 2 to 5. In theses tables, $N$ denotes the order of the discretization, i.e., the representable points are the numbers, $\frac{j}{N}$, with $0 \leq j < N$.

It seems that the computation of numerical approximations of the periodic orbits leads to unpredictable results.

Many more examples could be given, but those given may serve to illustrate the intriguing character of the results: the outcomes prove to be extremely sensitive to

| $N = 2^{23} = 8,388,608$ 7 cycles | | |
|---|---|---|
| period | basin size | relative size |
| 4,898 | 5,441,432 | 64.85 % |
| 1,746 | 2,946,734 | 35.13 % |
| 13 | 205 | $24.10^{-4}$ % |
| 6 | 132 | $16.10^{-4}$ % |
| 30 | 96 | $11.10^{-4}$ % |
| 4 | 8 | $< 1.10^{-6}$ % |
| 1 | 1 | $< 1.10^{-6}$ % |

Table 2: Coexisting periodic orbits of mapping (4) for the discretization $N = 2^{23}$.

| $N = 2^{24} = 16,777,216$ 2 cycles | | |
|---|---|---|
| period | basin size | relative size |
| 5,300 | 16,777,214 | 100 % |
| 1 | 2 | $< 1.10^{-6}$ % |

Table 3: Coexisting periodic orbits of mapping (4) for the discretization $N = 2^{24}$.

the details of the experiment, but the results all have a similar flavour: a relatively small number of cycles attracts near all orbits, and the lengths of these significant cycles are much larger than one but much smaller than the number of representable points. O. E. Lanford III wrote that there are here regularities which ought to be understood.

In [1], P. Diamond and A. Pokrovskii, suggest that statistical properties of the phenomenon of computational collapse of discretized chaotic mapping can be modeled by random mappings with an absorbing centre. The model gives results which are very much in line with computational experiments and there appears to be a type of universality summarised by an Arcsine law. The effects are discussed with special reference to the family of mappings

$$x_{n+1} = 1 - |1 - 2x_n|^{\ell} \quad 0 \le x \le 1 \quad 1 \le \ell \le 2 \tag{5}$$

Computer experiments show close agreement with prediction of the model.

However these results are of statistical nature, they do not give accurate information on the exact nature of the orbits (e.g. length of the shortest one, of the

| $N = 2^{24} - 1 = 16,777,215$ | | |
|---|---|---|
| 10 cycles | | |
| period | basin size | relative size |
| 3,081 | 7,502,907 | 44.72 % |
| 699 | 3,047,369 | 18.16 % |
| 3,469 | 2,905,844 | 17.32 % |
| 1,012 | 2,774,926 | 16.54 % |
| 563 | 290,733 | 11.73 % |
| 2,159 | 221,294 | 1.32 % |
| 138 | 21,610 | 0.13 % |
| 421 | 12,477 | 0.07 % |
| 9 | 54 | $< 1.10^{-3}$ % |
| 1 | 1 | $< 1.10^{-7}$ % |

Table 4: Coexisting periodic orbits of mapping (4) for the discretization $N = 2^{24} - 1$.

| $N = 2^{25} = 33,554,432$ | | |
|---|---|---|
| 8 cycles | | |
| period | basin size | relative size |
| 4,094 | 32,114,650 | 95.71 % |
| 621 | 918,519 | 2.74 % |
| 283 | 516,985 | 1.54 % |
| 126 | 2,937 | $< 0.01$ % |
| 6 | 887 | $< 0.01$ % |
| 55 | 433 | $< 0.01$ % % |
| 4 | 20 | $< 1.10^{-6}$ % |
| 1 | 1 | $< 1.10^{-6}$ % |

Table 5: Coexisting periodic orbits of mapping (4) for the discretization $N = 2^{25}$.

greater one, size of their basin of attraction). It is why we consider the problem of computational discrepancies in an original way in the next section.

# 3 Global orbit pattern

We introduce in this section a new mathematical tool: the global orbit pattern of a function, in order to describe more precisely which kind of behaviour occurs in discretized dynamical systems on finite sets.

## 3.1 Definitions

Let $f$ be a map from $X$ onto $X$.
We denote by $\forall x \in X,\ \ f^0(x) = x$ and $\forall p \geq 1, \forall x \in X,\quad f^p(x) = \underbrace{f \circ f \circ \ldots \circ f}_{p \quad \text{times}}(x)$.
Given $x_0 \in X$, we define the sequence $\{x_i\}$ of the associated dynamical system by $x_{i+1} = f(x_i)$ for $i \geq 0$. Thus $x_i = f^i(x_0)$.
The orbit of $x_0$ under $f$ is the set of points $\mathcal{O}(x_0, f) = \{f^i(x_0), i \geq 0\}$.
The starting point $x_0$ for the orbit is called the initial value of the orbit.
A point $x$ is a fixed point of the map $f$ if $f(x) = x$.
A point $x$ is a periodic point with period $p$ if $f^p(x) = x$ and $f^k(x) \neq x$ for all $k$ such that $0 \leq k < p$, $p$ is called the order of $x$.
If $x$ is periodic of order $p$, then the orbit of $x$ under $f$ is the finite set $\{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$. We will call this set the periodic orbit of order $p$ or a $p$-cycle.
A fixed point is then a 1-cycle.
The point $x$ is an eventually periodic point of $f$ with order $p$ if there exists $K > 0$ such that $\forall k \geq K\ f^{k+p}(x) = f^k(x)$.
$\forall x \in X$, we denote $\omega(x, f)$ the order of $x$ under $f$ or simply $\omega(x)$ when the map $f$ involved is obvious.
A subset $T$ of $X$ is invariant under $f$ if $f^{-1}(T) = T$. That is equivalent to say that $T$ is invariant under $f$ if and only if $f(T) \subset T$ and $f^{-1}(T) \subset T$.

## 3.2 Map on finite set

Along this paper, $N$ is a non-zero integer and $\sharp A$ stands for the cardinal of any finite set $A$. In this article, we consider $X$ as an ordered finite set with $N$ elements. We denote it $X_N$ and $X_N$ is isomorphic to the interval in $[\![0, N-1]\!]$ belonging to $\mathbb{N}$. Then $\sharp X_N = N$. Let $f$ be a map from $X_N$ into $X_N$. We denote by $\mathcal{F}_N$ the set of the maps from $X_N$ into $X_N$. Clearly, $\mathcal{F}_N$ is a finite set and $\sharp\mathcal{F}_N = N^N$ elements. For all

$x \in X_N$, $\mathcal{O}(x, f)$ is necessarily a finite set with at most $N$ elements. Indeed, let us consider the sequence $\{x, f(x), f^2(x), \ldots, f^{N-1}(x), f^N(x)\}$ of the first $N+1$ iterated points. Thanks to the Dirichlet's box principle, two elements are equals because $X_N$ has exactly $N$ different values. Thus, every initial value of $X_N$ leads ultimately to a repeating cycle. More precisely, if $x$ is a fixed point $\mathcal{O}(x, f)$ is the unique element $x$ and if $x$ is a periodic point with order $p$, $\mathcal{O}(x, f)$ has exactly $p$ elements. In this case, the orbit of $x$ under $f$ is the set $\mathcal{O}(x, f) = \{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$. If $x$ is an eventually periodic point with order $p$, there exists $K > 0$ such that $\forall k \geq K \ f^{k+p}(x) = f^k(x)$. In this case, the orbit of $x$ under $f$ is the set $\mathcal{O}(x, f) = \{x, f(x), f^2(x), \ldots, f^K(x), f^{K+1}(x), \ldots, f^{K+p-1}(x)\}$.

## 3.3 Equivalence classes

### 3.3.1 Components

Let $f \in \mathcal{F}_N$. We consider on $X_N$ the relation $\sim$ defined by : $\forall x, x' \in X_N, \ x \sim x' \Leftrightarrow \exists k \in \mathbb{N}$ such that $f^k(x) \in \mathcal{O}(x', f)$. The relation $\sim$ is an equivalence relation on $X_N$. $\mathcal{S}_N / \sim$ is the collection of the equivalence classes that we will call components of $X_N$ under $f$ which constitute a partition of $X_N$. The number of components are given in [3]. Asymptotic properties of the number of cycles and components are studied in [8]. For each component, we take as class representative element the least element of the component. The components will be written $T_N(x_0, f), \ldots, T_N(x_{p_{f,N}}, f)$ where $x_i$ is the least element of $T_N(x_i, f)$.

By analogy with real dynamical systems, we can define attractive and repulsive components in discretized dynamical systems as follows.

***Definition* 1** A component is repulsive when it is a cycle. Otherwise, the component is attractive.

***Remark*** In other words, a component is attractive when the component contains at least an eventually periodic element. The corresponding cycle is strictly contained in an attractive component.

Examples are given in the tables 6, 7 and 8.

For instance, in the table 7, the function $f$ has $\{8\}$ as orbit and $\{1, 7, 8\}$ as component which is attractive because $\{1\}$ and $\{7\}$ are eventually periodic elements.

| Function | | | orbit/component/nature | | |
|---|---|---|---|---|---|
| 0 | $\to$ | 4 | | | |
| 1 | $\to$ | 1 | $\{0,4\}$ | $\{0,4,3\}$ | attractive |
| 2 | $\to$ | 5 | | | |
| 3 | $\to$ | 4 | $\{1\}$ | $\{1\}$ | repulsive |
| 4 | $\to$ | 0 | | | |
| 5 | $\to$ | 9 | $\{5,9,7\}$ | $\{2,5,9,7\}$ | attractive |
| 6 | $\to$ | 8 | | | |
| 7 | $\to$ | 5 | $\{6,8\}$ | $\{6,8\}$ | repulsive |
| 8 | $\to$ | 6 | | | |
| 9 | $\to$ | 7 | | | |

Table 6: Orbits and components of a function belonging to $\mathcal{F}_{10}$ with gop $[2,1,3,2]_{10}$.

| Function | | | orbit/component/nature | | |
|---|---|---|---|---|---|
| 0 | $\to$ | 6 | | | |
| 1 | $\to$ | 8 | $\{0,6\}$ | $\{0,6\}$ | repulsive |
| 2 | $\to$ | 4 | | | |
| 3 | $\to$ | 9 | $\{8\}$ | $\{1,7,8\}$ | attractive |
| 4 | $\to$ | 5 | | | |
| 5 | $\to$ | 2 | $\{2,4,5\}$ | $\{2,4,5\}$ | repulsive |
| 6 | $\to$ | 0 | | | |
| 7 | $\to$ | 1 | $\{3,9\}$ | $\{3,9\}$ | repulsive |
| 8 | $\to$ | 8 | | | |
| 9 | $\to$ | 3 | | | |

Table 7: Orbits and components of another function belonging to $\mathcal{F}_{10}$ with gop $[2,1,3,2]_{10}$.

### 3.3.2 Order of elements

Here are some remarks on the order of elements of components.

***Remark*** The order of every element of a component is the length of its inner cycle.

| Function | | | orbit/component/nature | | |
|---|---|---|---|---|---|
| 0 | $\to$ | 9 | | | |
| 1 | $\to$ | 6 | $\{2,9\}$ | $\{0,2,9\}$ | attractive |
| 2 | $\to$ | 9 | | | |
| 3 | $\to$ | 8 | $\{6\}$ | $\{1,6\}$ | attractive |
| 4 | $\to$ | 3 | | | |
| 5 | $\to$ | 7 | $\{3,8,4\}$ | $\{3,8,4\}$ | repulsive |
| 6 | $\to$ | 6 | | | |
| 7 | $\to$ | 5 | $\{5,7\}$ | $\{5,7\}$ | repulsive |
| 8 | $\to$ | 4 | | | |
| 9 | $\to$ | 2 | | | |

Table 8: Orbits and components of a third function belonging to $\mathcal{F}_{10}$ with gop $[2,1,3,2]_{10}$.

***Definition* 2** For all $x \in X_N$, there exists $i \in [\![0, p_{f,N}]\!]$ such that $x$ belongs to the component $T_N(x_i, f)$. Then $\omega(x, f)$ is equal to the order $\omega(x_i, f)$.

***Remark*** For all $i \in [\![0, p_{f,N}]\!]$, $T_N(x_i, f)$ is an invariant subset of $X_N$ under $f$.

In the example given in the table 6, the order of the element $\{5\}$ is 3, the order of the element $\{1\}$ is 1, the order of the element $\{8\}$ is 2. The elements $\{2\}$ and $\{5\}$ have the same order.

## 3.4 Definition of global orbit pattern

For each $f \in \mathcal{F}_N$, we can determine the components of $X_N$ under $f$. For each component, we determine the order of any element. Thus, for each $f \in \mathcal{F}_N$, we have a set of orders that we will denote $\Omega(f, N)$. Be given $f$, there exist $p_{f,N}$ components and $p_{f,N}$ representative elements such that $x_0 < x_1 < \ldots < x_{p_{f,N}}$.

For each $f \in \mathcal{F}_N$, the sequence $[\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}}); f]_{\mathcal{F}_N}$ with $x_0 < x_1 < \ldots < x_{p_{f,N}}$ will design the global orbit pattern of $f \in \mathcal{F}_N$.

We will write $gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}}); f]_{\mathcal{F}_N}$.

When the reference to $f \in \mathcal{F}_N$ is obvious, we will write shortly

$gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}})]_N$ or $gop(f) = [\omega(x_0), \omega(x_1), \ldots, \omega(x_{p_{f,N}})]$ .

For example, the same gop associated to the functions given in the tables 6, 7 and 8 is $[2, 1, 3, 2]_{10}$.

Another example is given in the table 9. In that example, we have $\omega(0) = 2$, $\omega(3) = 1$, $\omega(4) = 4$.

| Function | orbit/component/nature | | |
|---|---|---|---|
| $0 \rightarrow 1$ | | | |
| $1 \rightarrow 0$ | $\{0, 1\}$ | $\{0, 1, 2\}$ | attractive |
| $2 \rightarrow 0$ | | | |
| $3 \rightarrow 3$ | $\{3\}$ | $\{3\}$ | repulsive |
| $4 \rightarrow 5$ | | | |
| $5 \rightarrow 6$ | $\{4, 5, 6, 7\}$ | $\{4, 5, 6, 7\}$ | repulsive |
| $6 \rightarrow 7$ | | | |
| $7 \rightarrow 4$ | | | |

Table 9: Orbits and components of a function belonging to $\mathcal{F}_8$ with gop $[2, 1, 4]_8$.

***Definition* 3** The set of all the global orbit patterns of $\mathcal{F}_N$ is called $\mathcal{G}(\mathcal{F}_N)$.

For example, for $N = 4$, the set $\mathcal{G}(\mathcal{F}_4)$ is $\{[1]; [1, 1]; [1, 1, 1]; [1, 2]; [1, 1, 1, 1]; [1, 1, 2]; [1, 2, 1]; [1, 3]; [2]; [2, 1]; [2, 1, 1]; [2, 2]; [3]; [3, 1]; [4]\}$.

## 3.5 Class of gop

We give the following definitions :

***Definition* 4** Let be $A = [\omega_1, \ldots, \omega_p]_N$ a gop. Then the class of $A$, written $\overline{A}$, is the set of all the functions $f \in \mathcal{F}_N$ such that the global orbit pattern associated to $f$ is $A$.

For example, for $N = 10$, the class of the gop $\overline{[2, 1, 3, 2]_{10}}$ contains the following few of many functions defined in the tables 6, 7 and 8. The periodic orbit which are encountered have the same length nevertheless there are different.

***Definition* 5** Let be $A = [\omega_1, \ldots, \omega_p]_N$ a gop.
Then the modulus of $A$ is $|A| = \sum_{k=1}^{p} \omega_k$.

***Remark*** $|[\omega_1, \ldots, \omega_p]_N| \leq N$.

## 3.6 Threshold functions

### 3.6.1 Ordering the discrete maps

***Theorem* 1** The sets $\mathcal{F}_N$ and $[\![1, N^N]\!]$ are isomorphic.

*Proof.* We define the function $\phi$ from $\mathcal{F}_N$ to $[\![1, N^N]\!]$ by : for each $f \in \mathcal{F}_N$, $\phi(f)$ is the integer $n$ such that $n = \sum_{k=0}^{N-1} f(k)N^{N-1-k} + 1$.
Then $\phi$ is well defined because $n \in [\![1, N^N]\!]$.
Let $n$ be a given integer between 1 and $N^N$. We convert $n-1$ in base $N$ : there exists a unique $N$-tuple $(a_{n-1,0}; a_{n-1,1}; \ldots; a_{n-1,N-1}) \in [\![0, N-1]\!]^N$ such that $\overline{n-1}^N = \sum_{i=0}^{N-1} a_{n-1,N-1-i}N^{N-i-1}$. We can thus define the map $f_n$ with : $\forall i \in X_N$, $f_n(i) = a_{n-1,N-i-1}$. Then $\phi$ is one to one.

***Remark*** This implies $\mathcal{F}_N$ is totally ordered.

***Definition* 6** Let $f \in \mathcal{F}_N$. Then $n = \sum_{k=0}^{N-1} f(k)N^{N-1-k} + 1$ is called the rank of $f$.

### 3.6.2 Threshold functions

Be given a global orbit pattern $A$, we are exploring the class $\overline{A}$.

***Theorem* 2** For every $A \in \mathcal{G}(\mathcal{F}_N)$, the class $\overline{A}$ has a unique function with minimal rank.

***Definition* 7** For every class $\overline{A} \in \mathcal{G}(\mathcal{F}_N)$, the function defined by the previous theorem will be called the threshold function for the class $\overline{A}$ and will be denoted by

$Tr(A)$ or $Tr(\overline{A})$.

To prove the theorem, we need the following definition :

***Definition* 8** Let $f \in \mathcal{F}_N$ be a function. Let $p$ be a non zero integer smaller than $N$. Let be $x_1, \dots, x_p$ $p$ consecutive elements of $X_N$. Then $x_1, \dots, x_p$ is a canonical $p$-cycle in relation to $f$ if $\forall j \in [\![1, p-1]\!]$ , $f(x_j) = x_{j+1}$ and $f(x_p) = x_1$.

*Proof.* Let $[\omega_1, \dots, \omega_p]$ be a global orbit pattern of $\mathcal{G}(\mathcal{F}_N)$. We construct a specific function $f$ belonging to the class $\overline{[\omega_1, \dots, \omega_p]}$ and we prove that the function so obtained is the smallest with respect to the order on $\mathcal{F}_N$. With the first $\omega_1$ elements of $[\![0, N-1]\!]$, that is the set of integers $[\![0, \omega_1 - 1]\!]$, we construct the canonical $\omega_1$-cycle: if $\omega_1 = 1$, we define $f(0) = 0$, else $f(0) = 1$, $f(1) = 2$, ..., $f(\omega_1 - 2) = \omega_1 - 1$, $f(\omega_1 - 1) = 0$.
Then $\forall j \in [\![\omega_1 - 1, \omega_1 + N - s - 1]\!]$, we define $f(j) = 0$.
Then with the next $\omega_2$ integers $[\![\omega_1 + N - s, \omega_1 + N - s + \omega_2 - 1]\!]$ we construct the canonical $\omega_2$-cycle. We keep going on constructing for all $j \in [\![3, p]\!]$ the canonical $\omega_j$-cycle.
In consequence, we have found a function $f$ belonging to the class $\overline{[\omega_1, \dots, \omega_p]}$.
Assume there exists a function $g \in \mathcal{F}_N$ belonging to the class of $f$ such that $g < f$. Let $I = \{i \in [\![0, N-1]\!] \text{ such that } f(i) \neq 0\}$. As $g < f$, there exists $i_0 \in I$ such that $g(i_0) < f(i_0)$. There exists also $j_0$ such that $i_0 \in \omega_{j_0}$. If $f(i_0) = i_0$, then $\omega_{j_0} = 1$, $g(i_0) < i_0$ and then $g(i_0) \notin \omega_{j_0}$. Then the global orbit pattern of $g$ does not contain anymore 1 as cycle. The global orbit pattern of $g$ is different from the global orbit pattern of $f$. If $f(i_0) = i_0 + 1$, then $g(i_0) \leq i_0$. Either $g(i_0) = i_0$ and then the global orbit pattern of $g$ is changed, or $g(i_0) < i_0$ and we are in the same situation as previously. Thus, in any case, the smallest function belonging to the class $\overline{[\omega_1, \dots, \omega_p]}$ is the one constructed in the first part of the proof.

The proof of the theorem gives an algorithm of construction of the threshold function associated to a given gop.
The threshold function associated to the gop $[2, 1, 3, 2]_{10}$ is explained in the table 10. Its rank is $n = 1,000,467,598$.

***Theorem* 3** There are exactly $2^N - 1$ different global orbit patterns in $\mathcal{F}_N$.

That is $\sharp\mathcal{G}(\mathcal{F}_N) = 2^N - 1$.

For example, for $N = 4$, $\sharp\mathcal{G}(\mathcal{F}_4) = 2^4 - 1 = 15$.

| First step | Second step | Third step | Fourth step | Fifth step |
|---|---|---|---|---|
| Construction of the first canonical 2-cycle | Construction of the last canonical 2-cycle | Construction of the canonical 3-cycle | Construction of the canonical 1-cycle | Filling the remaining images with 0 |
| $0 \to 1$ | $0 \to 1$ | $0 \to 1$ | $0 \to 1$ | $0 \to 1$ |
| $1 \to 0$ | $1 \to 0$ | $1 \to 0$ | $1 \to 0$ | $1 \to 0$ |
| $2 \to$ | $2 \to$ | $2 \to$ | $2 \to$ | $2 \to 0$ |
| $3 \to$ | $3 \to$ | $3 \to$ | $3 \to$ | $3 \to 0$ |
| $4 \to$ | $4 \to$ | $4 \to$ | $4 \to 4$ | $4 \to 4$ |
| $5 \to$ | $5 \to$ | $5 \to 6$ | $5 \to 6$ | $5 \to 6$ |
| $6 \to$ | $6 \to$ | $6 \to 7$ | $6 \to 7$ | $6 \to 7$ |
| $7 \to$ | $7 \to$ | $7 \to 5$ | $7 \to 5$ | $7 \to 5$ |
| $8 \to$ | $8 \to 9$ | $8 \to 9$ | $8 \to 9$ | $8 \to 9$ |
| $9 \to$ | $9 \to 8$ | $9 \to 8$ | $9 \to 8$ | $9 \to 8$ |

Table 10: Algorithm for the threshold function construction for the gop $[2, 1, 3, 2]_{10}$.

*Proof.* Let $p$ be an integer between 1 and $N$. Consider the set $L(p, N)$ of $p$-tuples $(\alpha_1, \ldots, \alpha_p) \in (\mathbb{N}^*)^p$ such that $\alpha_1 + \ldots + \alpha_p \leq N$.
We write $L(N) = \{L(p, N), p = 1 \ldots N\}$. $L(N)$ and $\mathcal{G}(\mathcal{F}_N)$ have the same elements. Then

$$\sharp\mathcal{G}(\mathcal{F}_N) = \sum_{p=1}^{p=N} \sharp L(p, N) = \sum_{p=1}^{p=N} \left( \begin{array}{c} N \\ p \end{array} \right) = 2^N - 1.$$

### 3.6.3 Ordering the global orbit patterns

We define an order relation on $\mathcal{G}(\mathcal{F}_N)$.

***Proposition* 1** Let $A$ and $B$ be two global orbit patterns of $\mathcal{G}(\mathcal{F}_N)$. We define the relation $\prec$ on the set $\mathcal{G}(\mathcal{F}_N)$ by

$$A \prec B \quad \textit{iff} \quad Tr(A) < Tr(B)$$

Then the set $(\mathcal{G}(\mathcal{F}_N), \prec)$ is totally ordered.

*Proof.* As the order $\prec$ refers to the natural order of $\mathbb{N}$, the proof is obvious.

Let $r \geq 1$, $p \geq 1$ be two integers. Let $[\omega_1, \ldots, \omega_p]$ and $[\omega'_1, \ldots, \omega'_r]$ be two global orbit patterns of $\mathcal{G}(\mathcal{F}_N)$. For example, if $p < r$, in order to compare them, we admit that we can fill $[\omega_1, \ldots, \omega_p]$ with $r-p$ zeros and write $[\omega_1, \ldots, \omega_p] = [\omega_1, \ldots, \omega_p, 0, \ldots, 0]$.

***Proposition* 2** Let $r \geq 1$, $p \geq 1$ be two integers such that $p \leq r$. Let $A = [\omega_1, \ldots, \omega_p]$ and $B = [\omega'_1, \ldots, \omega'_r]$ be two global orbit patterns.

- If $r = p = 1$ and $\omega_1 < \omega'_1$ then $A \prec B$.
- If $r \geq 2$ then
  - ∗ If $\omega_1 < \omega'_1$ then $A \prec B$.
  - ∗ If $\omega_1 = \omega'_1$ then there exists $K \in [\![2; r]\!]$ such that $\omega_K \neq \omega'_K$ and $\forall i < K$, $\omega_i = \omega'_i$.
    - If $|A| < |B|$, then $A \prec B$.
    - If $|A| = |B|$, then if $\omega_K < \omega'_K$ then $A \prec B$.

For example, for $N = 4$, the global orbit patterns are in increasing order : $[1] \prec [1,1] \prec [1,1,1] \prec [1,2] \prec [1,1,1,1] \prec [1,1,2] \prec [1,2,1] \prec [1,3] \prec [2] \prec [2,1] \prec [2,1,1] \prec [2,2] \prec [3] \prec [3,1] \prec [4]$.

#### 3.6.4 Algorithm for ordering the global orbit patterns : a pseudo-decimal order

The table 11 gives a method for ordering the gop : indeed, we consider each gop as if each one represents a decimal number: we begin to order them in considering the first order $\omega_1$. Considering two gops $A = [\omega_1, ..., \omega_p]$ and $A' = [\omega'_1, ..., \omega'_r]$, if $\omega_1 < \omega'_1$, then $A \prec A'$. For example, $[1,3] \prec [2,1]$. If $\omega_1 = \omega'_1$ and $|A| - \omega_1 < |A'| - \omega'_1$, then $A \prec A'$. For example to compare the gop $[1,2]$ and the gop $[1,1,1,1]$, we say that the first order $\omega_1$ stands for the unit digit - which is $\omega_1 = 1$ here, then the decimal digits are respectively 0.2 and 0.111. We calculate for each of them the modulus-$\omega_1$: we find $|[1,2]| - 1 = 2$ and $|[1,1,1,1]| - 1 = 3$, thus $[1,2] \prec [1,1,1,1]$. Finally, if $\omega_1 = \omega'_1$ and $|A| - \omega_1 = |A'| - \omega'_1$, then also we use the order of the decimal part. For example, $[1,1,1] \prec [1,2]$ because $1.11 < 1.2$. Applying this process, we have the sequence of the ordered gop for $N = 4$ given in the previous paragraph.

For example, for $N = 4$, we construct a tree (Figure 2) : each vertex is an ordered

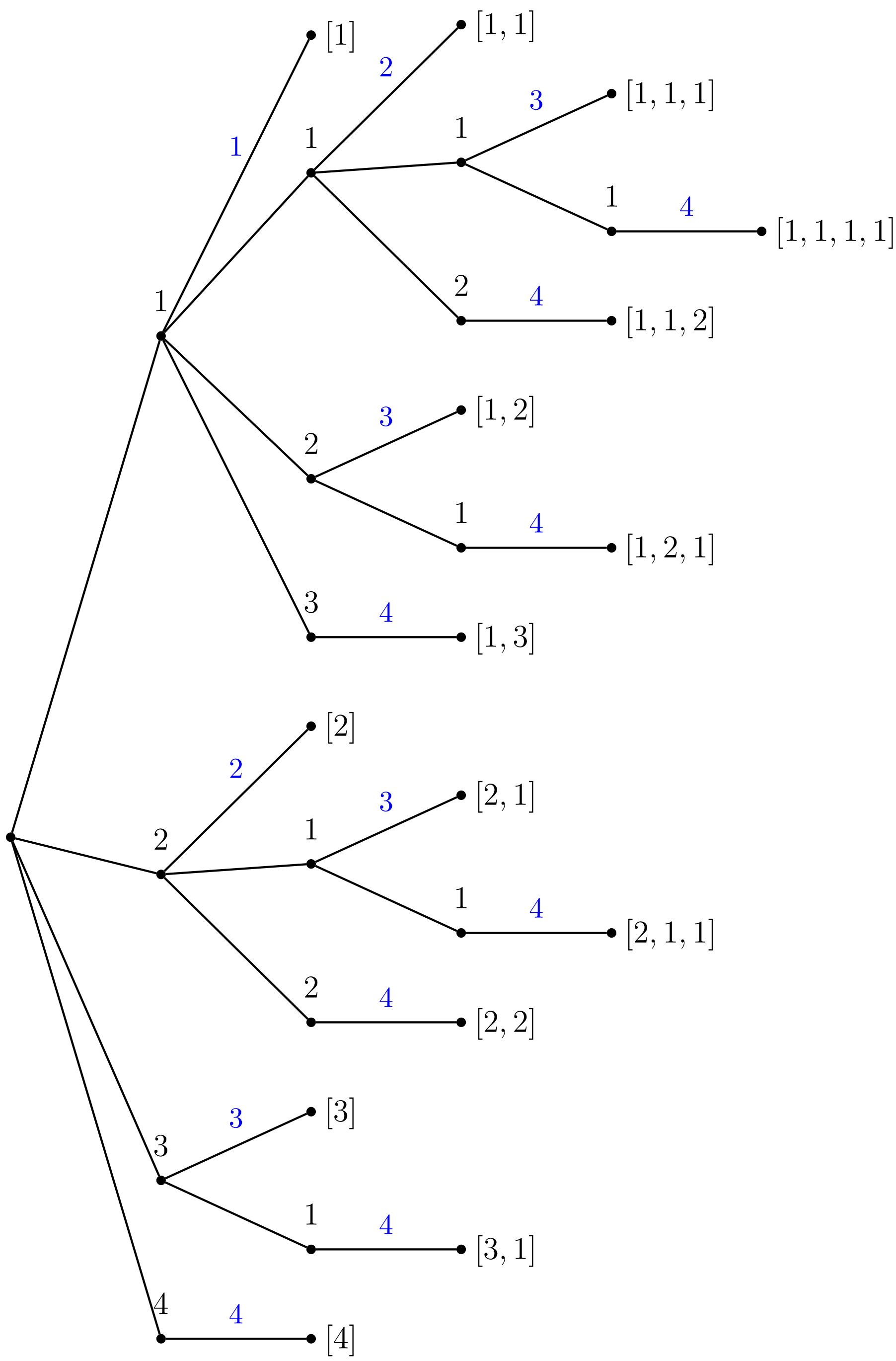


Figure 2: Tree for the construction of the order on $\mathcal{G}(\mathcal{F}_4)$

| Gop | Modulus | Modulus-$\omega_1$ | Gop | Modulus | Modulus-$\omega_1$ |
|---|---|---|---|---|---|
| $[1]$ | 1 | 0 | $[2]$ | 2 | 0 |
| $[1,1]$ | 2 | 1 | $[2,1]$ | 3 | 1 |
| $[1,1,1]$ | 3 | 2 | $[2,1,1]$ | 4 | 2 |
| $[1,2]$ | 3 | 2 | $[2,2]$ | 4 | 2 |
| $[1,1,1,1]$ | 4 | 3 | | | |
| $[1,1,2]$ | 4 | 3 | $[3]$ | 3 | 0 |
| $[1,2,1]$ | 4 | 3 | $[3,1]$ | 4 | 1 |
| $[1,3]$ | 4 | 3 | $[4]$ | 4 | 0 |

Table 11: Ordered gop for $N = 4$ with modulus and modulus-$\omega_1$

orbit, the modulus of the gop is written on the last edge.
However, the sequence of ordered gop differs from the natural downward lecture of the tree and has to be done following the algorithm.

# 4 Numbering the discrete maps

In this section we emphasize some closed formulas giving the cardinal of classes of gop. Recalling first the already known formula for the class $[\underbrace{1,\ldots,1}_{k \text{ times}}]_N$ for which we give a detailed proof, we consider the case were the class possesses exactly one $k$-cycle, the case with only two cycles belonging to the class and finally the main general formula of any cycles with any length. We give rigorous proof of all. The general formula is very interesting in the sense that even using computer network it is impossible to check every function of $\mathcal{F}_N$ when $N$ is larger than 100 or may be 50.

## 4.1 Discrete maps with 1-cycle only

The theorem 4 gives the number of discrete maps of $\mathcal{F}_N$ which have only fixed points and no cycles of length greater than one. This formula is explicit in [2] and [9]. A complete proof is given here in detail.

***Theorem* 4** Let $k$ be an integer between 1 and $N$. The number of functions whose global orbit pattern is $[\underbrace{1,\ldots,1}_{k \text{ times}}]_N$ (i.e. belonging to the class $[\underbrace{1,\ldots,1}_{k \text{ times}}]_N$) is

$\begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.

That is $\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.

*Proof.* Let $k$ be a non-zero integer. Let $f$ be a function of $\mathcal{F}_N$. There are $\begin{pmatrix} N \\ k \end{pmatrix}$ possibilities to choose $k$ fixed points. There remain $N-k$ points. Let $p$ be an integer between 1 and $N-k$. We assume that $p$ points are directly connected to the $k$ fixed points. For each of them, there are $k$ manners to choose one fixed point. There are $k^p$ ways to connect directly $p$ points to $k$ fixed points. There remains $N-k-p$ points that we must connect to the $p$ points. There are $\sharp[\underbrace{1,\ldots,1}_{p \text{ times}}]_{N-k}$ functions. Finally, the number of functions with $k$ fixed points is $\binom{N}{k} \sum_{p=1}^{N-k} k^p \ \sharp[\underbrace{1,\ldots,1}_{p \text{ times}}]_{N-k}$. We now prove recursively on $N$ for every $0 \le k \le N$ that $\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.
We have $\sharp[\overline{1}]_1 = 1$. The formula is true.
We suppose that $\forall k \le N$ $\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_N = \begin{pmatrix} N-1 \\ N-k \end{pmatrix} N^{N-k}$.
Let $X$ be a set with $N+1$ elements. We look for the functions of $\mathcal{F}_{N+1}$ which have $k$ fixed points. Thanks to the previous reasoning, we have

$$\begin{aligned}
\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_{N+1} &= \begin{pmatrix} N+1 \\ k \end{pmatrix} \sum_{p=1}^{N+1-k} k^p \ \sharp[\underbrace{1,\ldots,1}_{p \text{ times}}]_{N+1-k}. \\
\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_{N+1} &= \begin{pmatrix} N+1 \\ k \end{pmatrix} \sum_{p=1}^{N+1-k} k^p \ \sharp[\underbrace{1,\ldots,1}_{p \text{ times}}]_{N-(k-1)}.
\end{aligned}$$

We use the recursion assumption.

$$\begin{aligned}
\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_{N+1} &= \binom{N+1}{k} \sum_{p=1}^{N+1-k} k^p \binom{N-k}{p-1} (N-k+1)^{N-k+1-p}. \\
\sharp[\underbrace{1,\ldots,1}_{k \text{ times}}]_{N+1} &= k\binom{N+1}{k} \sum_{p=0}^{N-k} \begin{pmatrix} N-k \\ p \end{pmatrix} k^p (N-k+1)^{N-k-p}.
\end{aligned}$$

$$\sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_{N+1} = k\binom{N+1}{k}(N+1)^{N-k}$$

$$\sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_{N+1} = \binom{N}{k-1}(N+1)^{N-k+1}$$

$$\sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_{N+1} = \binom{N}{N-k+1}(N+1)^{N-k+1}.\text{q.e.d.}$$

## 4.2 Discrete maps with $k$-cycle

We look now for the number of functions with exactly one $k$-cycle.

***Theorem* 5** Let $k$ be an integer between 1 and $N$. The number of functions whose global orbit pattern is $[k]_N$ is $\sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_N \times (k-1)!$.

i.e. $\sharp[\overline{k}]_N = \sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_N \times (k-1)!$.

*Proof.* There are $\binom{N}{k}$ ways of choosing $k$ elements among $N$. Then, there are $(k-1)!$ choices for the image of those $k$ elements in order to constitute a $k$-cycle by $f$. We must now count the number of ways of connecting directly or not the remaining $N-k$ elements to the $k$-cycle. We established already this number which is equal to $\sum_{p=1}^{N-k} k^p \ \sharp[\underbrace{\boxed{1,\ldots,1}}_{p \text{ times}}]_{N-k}$. Finally, we have $\sharp[\overline{k}]_N = (k-1)!\binom{N}{k} \sum_{p=1}^{N-k} k^p \ \sharp[\underbrace{\boxed{1,\ldots,1}}_{p \text{ times}}]_{N-k}$. That is, $\sharp[k]_N = \sharp[\underbrace{\boxed{1,\ldots,1}}_{k \text{ times}}]_N \times (k-1)!$. q.e.d

## 4.3 Discrete maps with only two cycles

We give the number of functions with only two cycles.

***Theorem* 6** Let $N \geq 2$. Let $p$ and $q$ be two non-zero integers such that $p+q \leq N$. Then,
$\sharp[\overline{p,q}]_N = \sharp[\underbrace{\boxed{1,\ldots,1}}_{p+q \text{ times}}]_N \frac{(p+q-1)!}{q} = \frac{(N-1)! \ N^{N-(p+q)}}{(N-(p+q))! \ q}$.

*Proof.* We consider a function $f$ which belongs to the class $[\underbrace{1,\ldots,1}_{p+q \text{ times}}]_N$. We search the number of functions constructed from $f$ whose gop is $[p,q]_N$. From the $p$ fixed points of $f$, we construct a $p$-cycle. Thus, there are $\binom{p+q-1}{p-1}$ ways to choose $p-1$ integers among the $p+q-1$ fixed points. Counting the first given fixed point of $f$, we have $p$ points which allow to construct $(p-1)!$ functions with a $p$-cycle. Then there remain $q$ points which give $(q-1)!$ different functions with a $q$-cycle. Finally, the number of functions whose gop is $[p,q]_N$ is: $\binom{p+q-1}{p-1}(p-1)!(q-1)!$ that is the formula $\frac{(p+q-1)!}{q}$.

***Remark*** We notice that for all $k$ non-zero integer such that $k \leq N-1$, $\sharp\overline{[k,1]}_N = \sharp\overline{[k+1]}_N$.

## 4.4 Generalization : discrete maps with cycles of any length

We introduce now the main theorem of the section which gives the number of gop of discrete maps thanks to a closed formula.

Given a global orbit pattern $\alpha$, the next theorem gives a formula which gives the number of functions which belong to $\overline{\alpha}$.

***Theorem 7*** Let $p \geq 2$ be an integer. Let $[\omega_1,\ldots,\omega_p]_N$ be a gop of $\mathcal{G}(\mathcal{F}_N)$. Then,

$$\sharp\overline{[\omega_1,\ldots,\omega_p]}_N = \sharp\overline{[\underbrace{1,\ldots\ldots\ldots,1}_{\omega_1+\ldots+\omega_p \text{ times}}]}_N \frac{(\omega_1+\ldots+\omega_p-1)!}{\omega_p \times (\omega_{p-1}+\omega_p) \times \ldots \times (\omega_2+\ldots+\omega_p)},$$

$$\sharp\overline{[\omega_1,\ldots,\omega_p]}_N = \frac{(N-1)!\ N^{N-(\omega_1+\ldots+\omega_p)}}{(N-(\omega_1+\ldots+\omega_p))!\ \prod_{k=2}^{p}(\sum_{j=k}^{p}\omega_j)}$$

*Proof.* We consider a function $f$ which belongs to $\overline{[\underbrace{1,\ldots\ldots\ldots,1}_{\omega_1+\ldots+\omega_p \text{ times}}]}_N$. We search the number of functions constructed from $f$ whose gop is $[\omega_1,\ldots,\omega_p]_N$. From the $\omega_1$ fixed points of $f$, we construct a $\omega_1$-cycle. Thus, there are $\binom{\omega_1+\ldots+\omega_p-1}{\omega_1-1}$ ways to choose $\omega_1-1$ integers among the $\omega_1+\ldots+\omega_p-1$ fixed points. Counting the first given fixed point of $f$, we have $\omega_1$ points which allow to construct $(\omega_1-1)!$ functions with a $\omega_1$-cycle. Then, the first fixed point of $f$ which has not be chosen for the

$\omega_1$-cycle, will belong to the $\omega_2$-cycle. Thus, there are $\binom{\omega_2+\ldots+\omega_p-1}{\omega_2-1}$ ways to choose $\omega_2-1$ integers among the $\omega_2+\ldots+\omega_p-1$ fixed points. So we have $\omega_2$ points which allow to construct $(\omega_2-1)!$ functions with a $\omega_2$-cycle. We keep going on that way until there remain $\omega_p$ fixed points which allow to construct $(\omega_p-1)!$ functions with a $\omega_p$-cycle. Finally, we have constructed :
$\binom{\omega_1+\ldots+\omega_p-1}{\omega_1-1}(\omega_1-1)!\binom{\omega_2+\ldots+\omega_p-1}{\omega_2-1}(\omega_2-1)!\times\ldots\times\binom{\omega_{p-1}+\omega_p-1}{\omega_{p-1}-1}(\omega_{p-1}-1)!(\omega_p-1)!$ functions. We simplify and obtain the formula.

***Corollary* 1** Let $p$ be a non-zero integer. Let $[\omega_1,\ldots,\omega_p]_N$ be a gop of $\mathcal{G}(\mathcal{F}_N)$. We suppose that there exists $j$ such that $\omega_j\geq 2$. Let $h$ be an integer between 1 and $\omega_j-1$. Then
$\sharp\overline{[\omega_1,\ldots,\omega_j,\ldots,\omega_p]}_N=\sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N\times(h+\omega_{j+1}+\ldots+\omega_p)$.

*Proof.* $\sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N\times(h+\omega_{j+1}+\ldots+\omega_p)=\sharp\overline{[\underbrace{1,\ldots\ldots\ldots,1}_{\omega_1+\ldots+\omega_p\ \text{times}}]}_N$
$\times\ \frac{(\omega_1+\ldots+\omega_p-1)!(h+\omega_{j+1}+\ldots+\omega_p)}{\omega_p(\omega_{p-1}+\omega_p)\ldots(\omega_{j+1}+\ldots+\omega_p)(h+\omega_{j+1}+\ldots+\omega_p)(\omega_j+\omega_{j+1}+\ldots+\omega_p)\times\ldots\times(\omega_2+\ldots+\omega_p)}$.

We simplify and we exactly obtain

$$\sharp\overline{[\omega_1,\ldots,\omega_j-h,h,\omega_{j+1},\ldots,\omega_p]}_N\times(h+\omega_{j+1}+\ldots+\omega_p)=\sharp\overline{[\omega_1,\ldots,\omega_j,\ldots,\omega_p]}_N.$$

Examples :

$\sharp\overline{[2,1,3,2]}_{10}=302,400$.

$\sharp\overline{[5,2,10,8,15,2,3]}_{50}=29,775,702,147,667,389,218,762,343,520,975,006,348,329,578,044,480,000,000,000,000,000$.

$\sharp\overline{[5,2,10,8,15,2,3]}_{50}\cong 2.98\times 10^{63}$ among the $8.88\times 10^{84}$ functions of $\mathcal{F}_{50}$.

# 5 Functions with local properties

## 5.1 Functions with locally bounded range

Since several centuries, continuity and differentiability play a dramatic role in mathematical analysis. However these concepts are not transposable to the functions on finite sets. Then in this section, in order to obtain more precise results on the orbit of dynamical systems on finite sets, we introduce some subsets of $\mathcal{F}_N$, whose functions have local properties such as locally bounded range. In these subsets, the gop are found to be fully efficient in order to describe very precisely the dynamics of the orbits. We first consider the very simple subset $\mathcal{L}_{1,N}$ of functions for which the difference between $f(p)$ and $f(p+1)$ is drastically bounded. In subsection 5.2 we consider more sophisticated subsets.

We consider the set :

$$\mathcal{L}_{1,N} = \{f \in \mathcal{F}_N \text{ such that } \forall p, 0 \leq p \leq N-2, |f(p) - f(p+1)| \leq 1\}.$$

### 5.1.1 Orbits of $\mathcal{L}_{1,N}$

***Theorem* 8** If $f \in \mathcal{L}_{1,N}$ then $f$ has only periodic orbits of order 1 or 2.

*Proof.* We suppose that $f \in \mathcal{L}_{1,N}$ has a 3-cycle. We denote $(a; f(a); f^2(a))$ taking $a$ the smallest value of the 3-cycle. If $a < f(a) < f^2(a)$ then there exist two non-zero integers $e$ and $e'$ such that $f(a) = a + e$ and $f^2(a) = f(a) + e'$. Thus, $f^2(a) - e' \leq f^3(a) \leq f^2(a) + e'$. That is $f(a) \leq a \leq f(a) + 2e'$. And finally we have the relation $a + e \leq a$ which is impossible.
If $a < f^2(a) < f(a)$ then there exist two non-zero integers $e$ and $e'$ such that $f^2(a) = a + e$ and $f(a) = f^2(a) + e'$. Thus, $f(a) - e \leq f^3(a) \leq f(a) + e$. That is $f(a) - e \leq a \leq f(a) + e$. But $f(a) - e = a + e'$. And finally we have the relation $a + e' \leq a$ which is impossible.
We can prove in the same way that the function $f$ can't have either 3-cycle or greater order cycle than 3.

### 5.1.2 Numerical results and conjectures

We have done numerical studies of the $\mathcal{G}(\mathcal{L}_{1,N})$ for $N = 1$ to 16, using the brute force of a desktop computer (i.e. checking every function belonging to these sets).

The tables 12, 13, 14, 15, 16 and 17 show the sequences for $\mathcal{L}_{1,1}$ to $\mathcal{L}_{1,16}$.

In theses tables we display in the first column all the gop of $\mathcal{G}(\mathcal{L}_{1,N})$ for every value of $N$. For a given $N$, there are two columns; the left one displays the cardinal of every existing class of gop (- stands for non existing gop). Instead the second shows more regularity, displaying on the row of the gop $[\underbrace{2, 2, ..., 2}_{k \text{ times}}]$ the sum of the cardinals of all the classes of the gop of the form $[\underbrace{2, 2, \ldots\ldots, \underbrace{1}_{i^{\ th}}, \ldots, 2}_{k+1 \text{ orders}}]$ which exist.

Then we are able to formulate some statements which have not yet been proved.

***Statement* 1** $$\sharp[\underbrace{1, 1, \ldots\ldots, 1}_{N-k+1 \text{ times}}]_{\mathcal{L}_{1,N}} = \begin{cases} 1 & \text{if } \ k = 1 \\ 2 & \text{if } \ k = 2 \\ \left(\frac{4}{27}\right)(k+1) \times 3^k & \text{for } \ 3 \leq k \leq \frac{N+1}{2} \end{cases}$$

***Remark*** We call $u_k = \sharp[\underbrace{1, 1, \ldots\ldots, 1}_{N-k+1 \text{ times}}]_{\mathcal{L}_{1,N}}$. For $k > 2$, then $u_k$ is the sequence A120926 On-line Encyclopedia of integer Sequences : it is the number of sequences where 0 is isolated in ternary words of length $N$ written with $\{0, 1, 2\}$.

***Statement* 2** $\sharp[\underbrace{1, 1, \ldots\ldots, 1}_{k \text{ times}}]_{\mathcal{L}_{1,N}} = \sharp[\underbrace{1, 1, \ldots\ldots, 1}_{k+1 \text{ times}}]_{\mathcal{L}_{1,N+1}}$ for $k \leq \frac{N+1}{2}$.

***Statement* 3** $\sharp[\underbrace{2, 2, \ldots\ldots, 2}_{k \text{ times}}]_{\mathcal{L}_{1,N}} = \sharp[\underbrace{2, 2, \ldots\ldots, 2}_{k+1 \text{ times}}]_{\mathcal{L}_{1,N+2}}$ for $k \leq \frac{N}{2}$.

***Statement* 4**

$$\sharp[\underbrace{2, 2, \ldots\ldots, 2}_{k \text{ times}}]_{\mathcal{L}_{1,N}} = \sharp[\underbrace{2, 2, \ldots\ldots, 2}_{k \text{ times}}, 1]_{\mathcal{L}_{1,N+1}} \text{ for } 2k \leq N \leq 3k - 1.$$

***Statement* 5**

$$\sharp[\underbrace{2, 2, \ldots\ldots, 2}_{k \text{ times}}]_{\mathcal{L}_{1,N}} = \sum_{i=1}^{k+1} \sharp[\underbrace{2, 2, \ldots\ldots, \underbrace{1}_{i^{\ th}}, \ldots, 2}_{k+1 \text{ orders}}]_{\mathcal{L}_{1,N}} \text{ for } 2k + 1 \leq N.$$

These statements show that first the set $\mathcal{L}_{1,N}$ is an interesting set to be considered for dynamical systems and secondly the gop are fruitful in this study. However the set

$$\mathcal{L}_{2,N} = \{f \in \mathcal{F}_N \text{ such that } \forall p, 0 \leq p \leq N - 2, |f(p) - f(p+1)| \leq 2\}$$

is too much large to give comparable results. Then we introduce more sophisticated sets we call sets with locally bounded range which more or less correspond to

| g.o.p. | N=1 | N=1 | N=2 | N=2 | N=3 | N=3 | N=4 | N=4 |
|---|---|---|---|---|---|---|---|---|
| Total number | | 1 | | 4 | | 17 | | 68 |
| $[1]$ | 1 | + | 2 | + | 7 | + | 26 | + |
| $[1,1]$ | - | + | 1 | + | 4 | + | 14 | + |
| $[1,1,1]$ | - | + | - | + | 1 | + | 4 | + |
| $[1,1,1,1]$ | - | + | - | + | - | + | 1 | + |
| $[2]$ | - | + | 1 | 1 | 4 | 4 | 18 | 18 |
| $[2,1]$ | - | + | - | + | 1 | 1 | 3 | 4 |
| $[1,2]$ | - | + | - | + | - | + | 1 | + |
| $[2,2]$ | - | + | - | + | - | + | 1 | 1 |

Table 12: Numbering functions with local properties for $f \in \mathcal{L}_{1,1}$, $f \in \mathcal{L}_{1,2}$, $f \in \mathcal{L}_{1,3}$, $f \in \mathcal{L}_{1,4}$.

an analogue of the discrete convolution product of the local variation of $f$ with a compact support function $\overrightarrow{\alpha_t}$.

## 5.2 Orbits and gop of locally bounded range function sets

Consider now the set :

$\mathcal{L}_{\overrightarrow{\alpha_t},q,N} = \{f \in \mathcal{F}_N$ such that $\forall p, 0 \leq p \leq N-r-1, \sum_{r=1}^{r=t} \alpha_r |f(p) - f(p+r)| \leq q\} \bigcap \{f \in \mathcal{F}_N$ such that $\forall p, t \leq p \leq N-1, \sum_{r=1}^{r=t} \alpha_r |f(p) - f(p-r)| \leq q\}$ for the vector $\overrightarrow{\alpha_t} = (\alpha_1, \alpha_2, \ldots, \alpha_t) \in \mathbb{N}^t$, for $q \in \mathbb{N}$.

The functions belonging to these sets show a kind of "rigidity": the less is $q$, the more "rigid" is the function. Furthermore, the maximal length of a periodic orbit increases with $q$, and so the number of gop $\sharp\mathcal{G}(\mathcal{L}_{\overrightarrow{\alpha_t},q,N})$ and the maximal modulus of the gop. As an example, we explore numerically the case: $N = 10$, $t = 5$, $\alpha_1 = 20$, $\alpha_2 = 10$, $\alpha_3 = 5$, $\alpha_4 = 3$ and $\alpha_5 = 1$, for $q = 35, \ldots, 101$. The results are displayed in table 18. In this table "modulus" means the maximal modulus of the gop belonging to this set for the corresponding value of $q$ in the row, "gop number" stands for $\sharp\mathcal{G}(\mathcal{L}_{\overrightarrow{\alpha_t},q,N})$ and "functions number" for $\sharp\mathcal{L}_{\overrightarrow{\alpha_t},q,N}$. One can point out that for the particular function $\overrightarrow{\alpha_t}$ of the example; it is possible to find 10 intervals $I_1, I_2, \ldots, I_{10} \subset \mathbb{N}$ such that if $q \in I_r$ then there is no periodic orbit whose period is strictly greater than $r$, (e.g., $I_6 = [\![70, 82]\!]$). Furthermore it is possible to split these intervals into subintervals $I_{r,s}$ in which $\sharp\mathcal{G}(\mathcal{L}_{\overrightarrow{\alpha_t},q,N})$ is constant when $q$ thumbs $I_{r,s}$.

| g.o.p. | N=5 | N=5 | N=6 | N=6 | N=7 | N=7 |
|---|---|---|---|---|---|---|
| Total number | | 259 | | 950 | | 387 |
| $[1]$ | 95 | + | 340 | + | 1,193 | + |
| $[1,1]$ | 50 | + | 174 | + | 600 | + |
| $[1,1,1]$ | 16 | + | 58 | + | 204 | + |
| $[1,1,1,1]$ | 4 | + | 16 | + | 60 | + |
| $[1,1,1,1,1]$ | 1 | + | 4 | + | 16 | + |
| $[1,1,1,1,1,1]$ | - | + | 1 | + | 4 | + |
| $[1,1,1,1,1,1,1]$ | - | + | - | + | 1 | + |
| $[2]$ | 70 | 70 | 264 | 264 | 952 | 952 |
| $[2,1]$ | 12 | 18 | 45 | 70 | 166 | 264 |
| $[1,2]$ | 6 | + | 25 | + | 98 | + |
| $[2,2]$ | 4 | 4 | 18 | 18 | 70 | 70 |
| $[2,2,1]$ | 1 | 1 | 4 | 4 | 17 | 18 |
| $[1,2,2]$ | - | + | - | + | 1 | + |
| $[2,1,2]$ | - | + | - | + | - | + |
| $[2,2,2]$ | - | + | 1 | 1 | 4 | 4 |
| $[2,2,2,1]$ | - | + | - | + | 1 | 1 |

Table 13: Numbering functions with local properties for $f \in \mathcal{L}_{1,5}$, $f \in \mathcal{L}_{1,6}$, $f \in \mathcal{L}_{1,7}$.

| g.o.p. | N=8 | N=8 | N=9 | N=9 | N=10 | N=10 |
|---|---|---|---|---|---|---|
| Total number | | 11,814 | | 40,503 | | 13,6946 |
| [1] | 4,116 | + | 14,001 | + | 47,064 | + |
| $[1,1]$ | 2,038 | + | 6,852 | + | 22,806 | + |
| $[1,1,1]$ | 700 | + | 2,366 | + | 7,896 | + |
| $[1,1,1,1]$ | 214 | + | 742 | + | 2,520 | + |
| $[1,1,1,1,1]$ | 60 | + | 216 | + | 754 | + |
| $[1,1,1,1,1,1]$ | 16 | + | 60 | + | 216 | + |
| $[1,1,1,1,1,1,1]$ | 4 | + | 16 | + | 60 | + |
| $[1,1,1,1,1,1,1,1]$ | 1 | + | 4 | + | 16 | + |
| $[1,1,1,1,1,1,1,1,1]$ | - | + | 1 | + | 4 | + |
| $[1,1,1,1,1,1,1,1,1,1]$ | - | + | - | + | 1 | + |
| [2] | 3,356 | 3,356 | 11,580 | 11,580 | 39,364 | 39,364 |
| $[2,1]$ | 590 | 952 | 2,062 | 3,356 | 7,072 | 11,580 |
| $[1,2]$ | 362 | + | 1,294 | + | 4,508 | + |
| $[2,2]$ | 264 | 264 | 952 | 952 | 3,356 | 3,356 |
| $[2,2,1]$ | 62 | 70 | 222 | 264 | 770 | 952 |
| $[1,2,2]$ | 6 | + | 28 | + | 113 | + |
| $[2,1,2]$ | 2 | + | 14 | + | 69 | + |
| $[2,2,2]$ | 18 | 18 | 70 | 70 | 264 | 264 |
| $[2,2,2,1]$ | 4 | 4 | 18 | 18 | 69 | 70 |
| $[1,2,2,2]$ | - | + | - | + | 1 | + |
| $[2,1,2,2]$ | - | + | - | + | - | + |
| $[2,2,1,2]$ | - | + | - | + | - | + |
| $[2,2,2,2]$ | 1 | 1 | 4 | 4 | 18 | 18 |
| $[2,2,2,2,1]$ | - | + | 1 | 1 | 4 | 4 |
| $[1,2,2,2,2]$ | - | + | - | + | - | + |
| $[2,1,2,2,2]$ | - | + | - | + | - | + |
| $[2,2,1,2,2]$ | - | + | - | + | - | + |
| $[2,2,2,1,2]$ | - | + | - | + | - | + |
| $[2,2,2,2,2]$ | - | + | - | + | 1 | 1 |

Table 14: Numbering functions with local properties for $f \in \mathcal{L}_{1,8}$, $f \in \mathcal{L}_{1,9}$, $f \in \mathcal{L}_{1,10}$.

| g.o.p. | N=11 | N=11 | N=12 | N=12 | N=13 | N=13 |
|---|---|---|---|---|---|---|
| Total number | | 457,795 | | 1,515,926 | | 4,979,777 |
| $[1]$ | 156,629 | + | 516,844 | + | 1,693,073 | + |
| $[1,1]$ | 75,292 | + | 246,762 | + | 803,706 | + |
| $[1,1,1]$ | 26,098 | + | 85,556 | + | 278,580 | + |
| $[1,1,1,1]$ | 8,434 | + | 27,904 | + | 91,488 | + |
| $[1,1,1,1,1]$ | 2,756 | + | 8,658 | + | 28,738 | + |
| $[1,1,1,1,1,1]$ | 756 | + | 2,590 | + | 8,730 | + |
| $[1,1,1,1,1,1,1]$ | 216 | + | 756 | + | 2,592 | + |
| $[1,1,1,1,1,1,1,1]$ | 60 | + | 216 | + | 756 | + |
| $[1,1,1,1,1,1,1,1,1]$ | 16 | + | 60 | + | 216 | + |
| $[1,1,1,1,1,1,1,1,1,1]$ | 4 | + | 16 | + | 60 | + |
| $[1,1,1,1,1,1,1,1,1,1,1]$ | 1 | + | 4 | + | 16 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1]$ | - | + | 1 | + | 4 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1]$ | - | + | - | + | 1 | + |
| $[2]$ | 132,104 | 132,104 | 438,846 | 438,846 | 1,445,258 | 1,445,258 |
| $[2,1]$ | 23,941 | 39,364 | 80,108 | 132,104 | 265,548 | 438,846 |
| $[1,2]$ | 15,423 | + | 51,996 | + | 173,298 | + |
| $[2,2]$ | 11,580 | 11,580 | 39,364 | 39,364 | 132,104 | 132,104 |
| $[2,2,1]$ | 2,634 | 3,356 | 8,883 | 11,580 | 29,659 | 39,364 |
| $[1,2,2]$ | 429 | + | 1,555 | + | 5,478 | + |
| $[2,1,2]$ | 293 | + | 1,142 | + | 4,227 | + |
| $[2,2,2]$ | 952 | 952 | 3,356 | 3,356 | 11,580 | 11,580 |
| $[2,2,2,1]$ | 255 | 264 | 899 | 952 | 3,098 | 3,356 |
| $[1,2,2,2]$ | 7 | + | 35 | + | 152 | + |
| $[2,1,2,2]$ | 2 | + | 16 | + | 86 | + |
| $[2,2,1,2]$ | - | + | 2 | + | 20 | + |
| $[2,2,2,2]$ | 70 | 70 | 264 | 264 | 952 | 952 |
| $[2,2,2,2,1]$ | 18 | 18 | 70 | 70 | 263 | 264 |
| $[1,2,2,2,2]$ | - | + | - | + | 1 | + |
| $[2,1,2,2,2]$ | - | + | - | + | - | + |
| $[2,2,1,2,2]$ | - | + | - | + | - | + |
| $[2,2,2,1,2]$ | - | + | - | + | - | + |
| $[2,2,2,2,2]$ | 4 | 4 | 18 | 18 | 70 | 70 |
| $[2,2,2,2,2,1]$ | 1 | 1 | 4 | 4 | 18 | 18 |
| $[1,2,2,2,2,2]$ | - | + | - | + | - | + |
| $[2,2,2,2,2,2]$ | - | + | 1 | 1 | 4 | 4 |
| $[2,2,2,2,2,2,1]$ | - | + | - | + | 1 | 1 |

Table 15: Numbering functions with local properties for $f \in \mathcal{L}_{1,11}$, $f \in \mathcal{L}_{1,12}$, $f \in \mathcal{L}_{1,13}$.

| g.o.p. | N=14 | N=14 | N=15 | N=15 |
|---|---|---|---|---|
| Total number | | 16,246,924 | | 52,694,573 |
| [1] | 5,511,218 | + | 17,841,247 | + |
| [1, 1] | 2,603,258 | + | 8,391,360 | + |
| [1, 1, 1] | 901,802 | + | 2,904,592 | + |
| [1, 1, 1, 1] | 297,728 | + | 962,888 | + |
| [1, 1, 1, 1, 1] | 94,440 | + | 307,848 | + |
| [1, 1, 1, 1, 1, 1] | 29,050 | + | 95,676 | + |
| [1, 1, 1, 1, 1, 1, 1] | 8,746 | + | 29,140 | + |
| [1, 1, 1, 1, 1, 1, 1, 1] | 2,592 | + | 8,748 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1] | 756 | + | 2,592 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | 216 | + | 756 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | 60 | + | 216 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | 16 | + | 60 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | 4 | + | 16 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | 1 | + | 4 | + |
| [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] | - | + | 1 | + |
| [2] | 4,725,220 | 4,725,220 | 15,352,392 | 15,352,392 |
| [2, 1] | 873,149 | 1,445,258 | 2,851,350 | + |
| [1, 2] | 572,109 | + | 1,873,870 | + |
| [2, 2] | 438,846 | 438,846 | 1,445,258 | 1,445,258 |
| [2, 2, 1] | 98,135 | 132,104 | 322,310 | 438,846 |
| [1, 2, 2] | 18,873 | + | 63,967 | + |
| [2, 1, 2] | 15,096 | + | 52,569 | + |
| [2, 2, 2] | 39,364 | 39,364 | 132,104 | 132,104 |
| [2, 2, 2, 1] | 10,460 | 11,580 | 34,845 | 39,364 |
| [1, 2, 2, 2] | 605 | + | 2,282 | + |
| [2, 1, 2, 2] | 389 | + | 1,596 | + |
| [2, 2, 1, 2] | 126 | + | 641 | + |
| [2, 2, 2, 2] | 3,356 | 3,356 | 11,580 | 11,580 |
| [2, 2, 2, 2, 1] | 942 | 952 | 3,292 | 3,356 |
| [1, 2, 2, 2, 2] | 8 | + | 44 | + |
| [2, 1, 2, 2, 2] | 2 | + | 18 | + |
| [2, 2, 1, 2, 2] | - | + | 2 | + |
| [2, 2, 2, 1, 2] | - | + | - | + |
| [2, 2, 2, 2, 2] | 264 | 264 | 952 | 952 |
| [2, 2, 2, 2, 2, 1] | 70 | 70 | 264 | 264 |
| [1, 2, 2, 2, 2, 2] | - | + | - | + |
| [2, 2, 2, 2, 2, 2] | 18 | 18 | 70 | 70 |
| [2, 2, 2, 2, 2, 2, 1] | 4 | 4 | 18 | 18 |
| [2, 2, 2, 2, 2, 2, 2] | 1 | 1 | 4 | 4 |
| [2, 2, 2, 2, 2, 2, 2, 1] | - | + | 1 | 1 |

Table 16: Numbering functions with local properties for $f \in \mathcal{L}_{1,14}$, $f \in \mathcal{L}_{1,15}$.

| g.o.p. | N=16 | N=16 |
|---|---:|---:|
| Total number | | 170,028,792 |
| $[1]$ | 57,477,542 | + |
| $[1,1]$ | 26,932,398 | + |
| $[1,1,1]$ | 9,314,088 | + |
| $[1,1,1,1]$ | 3,097,650 | + |
| $[1,1,1,1,1]$ | 996,764 | + |
| $[1,1,1,1,1,1]$ | 312,456 | + |
| $[1,1,1,1,1,1,1]$ | 96,096 | + |
| $[1,1,1,1,1,1,1,1]$ | 29,158 | + |
| $[1,1,1,1,1,1,1,1,1]$ | 8,748 | + |
| $[1,1,1,1,1,1,1,1,1,1]$ | 2,592 | + |
| $[1,1,1,1,1,1,1,1,1,1,1]$ | 756 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1]$ | 216 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1]$ | 60 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1,1]$ | 16 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$ | 4 | + |
| $[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$ | 1 | + |
| $[2]$ | 49,610,818 | 49,610,818 |
| $[2,1]$ | 9,255,822 | 15,352,392 |
| $[1,2]$ | 6,096,570 | + |
| $[2,2]$ | 4,725,220 | 4,725,220 |
| $[2,2,1]$ | 1,051,686 | 1,445,258 |
| $[1,2,2]$ | 213,975 | + |
| $[2,1,2]$ | 179,597 | + |
| $[2,2,2]$ | 438,846 | 438,846 |
| $[2,2,2,1]$ | 114,798 | 132,104 |
| $[1,2,2,2]$ | 8,284 | + |
| $[2,1,2,2]$ | 6,146 | + |
| $[2,2,1,2]$ | 2,876 | + |
| $[2,2,2,2]$ | 39,364 | 39,364 |
| $[2,2,2,2,1]$ | 11,246 | 11,580 |
| $[1,2,2,2,2]$ | 204 | + |
| $[2,1,2,2,2]$ | 106 | + |
| $[2,2,1,2,2]$ | 22 | + |
| $[2,2,2,1,2]$ | 2 | + |
| $[2,2,2,2,2]$ | 3,356 | 3,356 |
| $[2,2,2,2,2,1]$ | 951 | 952 |
| $[1,2,2,2,2,2]$ | 1 | + |
| $[2,2,2,2,2,2]$ | 264 | 264 |
| $[2,2,2,2,2,2,1]$ | 70 | 70 |
| $[2,2,2,2,2,2,2]$ | 18 | 18 |
| $[2,2,2,2,2,2,2,1]$ | 4 | 4 |
| $[2,2,2,2,2,2,2,2]$ | 1 | 1 |

Table 17: Numbering functions with local properties for $f \in \mathcal{L}_{1,16}$.

| q | maximal period | modulus | gop number | functions number |
|---|---|---|---|---|
| 35 | 2 | 2 | 3 | 9,992 |
| 41 | 2 | 3 | 6 | 21,764 |
| 48 | 3 | 3 | 7 | 63,408 |
| 51 | 3 | 4 | 9 | 122,316 |
| 54 | 4 | 4 | 15 | 258,910 |
| 58 | 4 | 5 | 19 | 497,106 |
| 60 | 4 | 5 | 25 | 696,586 |
| 61 | 4 | 10 | 37 | 818,000 |
| 62 | 4 | 10 | 44 | 921,698 |
| 63 | 4 | 10 | 46 | 1,022,184 |
| 68 | 5 | 10 | 50 | 1,604,518 |
| 70 | 6 | 10 | 60 | 1,837,088 |
| 72 | 6 | 10 | 61 | 2,124,974 |
| 73 | 6 | 10 | 88 | 2,352,560 |
| 76 | 6 | 10 | 98 | 3,514,608 |
| 77 | 6 | 10 | 99 | 4,001,306 |
| 81 | 6 | 10 | 100 | 6,499,244 |
| 82 | 6 | 10 | 104 | 7,230,576 |
| 83 | 7 | 10 | 109 | 8,113,212 |
| 84 | 8 | 10 | 117 | 9,126,054 |
| 85 | 8 | 10 | 130 | 10,184,542 |
| 86 | 8 | 10 | 131 | 11,244,702 |
| 87 | 8 | 10 | 145 | 12,311,866 |
| 88 | 8 | 10 | 161 | 13,485,506 |
| 89 | 8 | 10 | 175 | 14,692,658 |
| 90 | 8 | 10 | 176 | 15,984,782 |
| 92 | 8 | 10 | 182 | 18,775,284 |
| 93 | 8 | 10 | 188 | 20,252,084 |
| 94 | 8 | 10 | 193 | 21,640,666 |
| 95 | 8 | 10 | 195 | 23,021,112 |
| 96 | 8 | 10 | 242 | 24,479,312 |
| 97 | 8 | 10 | 298 | 26,163,582 |
| 98 | 8 | 10 | 335 | 28,285,274 |
| 99 | 9 | 10 | 377 | 30,861,396 |
| 100 | 10 | 10 | 463 | 34,086,310 |
| 101 | 10 | 10 | 484 | 37,553,504 |

Table 18: Numerical study of the set $\mathcal{L}_{\vec{\alpha_t},q,N}$ for $N = 10$, $t = 5$, $\alpha_1 = 20$, $\alpha_2 = 10$, $\alpha_3 = 5$, $\alpha_4 = 3$ and $\alpha_5 = 1$, for $q = 35, ..., 101$

This is not the case for $\sharp\mathcal{L}_{\overrightarrow{\alpha_t},q,N}$.

# 6 Conclusion

The behaviour of a discrete dynamical system associated to a function on finite ordered set $X$ is not easily predictable. Such a system can only exhibit periodic orbits. All the orbits can be explicitly computed; all together they form a global orbit pattern. We formalise such a gop as the ordered set of periods when the initial value thumbs $X$ in the increasing order. We are able to predict, using closed formulas, the number of gop for the set $\mathcal{F}_N$ of all the functions on $X$. We introduce special subsets of $\mathcal{F}_N$ in order to understand more precisely the behaviour of the dynamical system. We explore by computer experiments theses sets, they show interesting patterns of gop. Further study is needed to understand the behaviour of dynamical systems associated to functions belonging to these sets.

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